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Four-dimensional kinks

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Abstract. The homotopy classes of Lorentz signature metric tensor fields on parallelisable four-manifolds are classified for both compact and non-compact manifolds. It is demonstrated that the homotopy classes form Abelian groups whose generators correspond to generalised elementary 'kinks', and their extension structures are analysed. It is also demonstrated that all odd kink states have the fermionic property of being odd under 360° rotations. Speculations on the physical interpretation of four-dimensional kink states are made which generalise those of Shastri, Williams and Zvengrowski from three to four dimensions.

1. Introduction

If a four-dimensional manifold X carries a Lorentz signature metric tensor field, the relativistic physics generated by the structure ought to be stable to small smooth deformations of the underlying manifold and the metric. By extending the range of deformation, the question arises as to what structure is left if arbitrary homotopies are allowed, that is, what are the purely topological invariants of a pair comprising a four-manifold and a tangent Lorentz metric, and what physical interpretation, if any, can be placed on these invariants? The classification of homotopy classes of Lorentz metrics on homotopy classes of four-manifolds is therefore of great interest in general relativity theory. Indeed, by associating characteristic algebraic invariants with homotopy classes, various authors, beginning with Finkelstein and Misner (1959), have been able to associate particle-like structures called 'kinks' with the homotopy classes.

Shastri *et al* (1980) considered the homotopy classes of Lorentz metrics on a distinguished class of non-compact four-manifolds which arise naturally in general relativity. These are the bundle spaces X of line bundles on compact, closed, parallelisable three-manifolds M, where X is homotopy equivalent to M. The above authors were able to establish that for this type of parallelisable four-manifold, the set of homotopy classes of Lorentz metrics is an Abelian group whose generators are identifiable with 'elementary' kinks. The latter group was shown to be a group extension of $H^3(M, Z)$ by $H^1(M, Z_2)$, and it was also shown that each kink state can be labelled by an integer 'kink number' which counts the elementary kinks. By displaying the group of homotopy classes as a group extension, one is able to display the elementary kink generators and list the various possible ways that the elementary kinks combine by listing the possible group extension structures. It was also shown that for certain M, odd kink states have the fermion-like property of 'changing sign' under 360° rotations about any spatial axis. The above authors suggested the following possible link between kinks and elementary particles. Suppose that an elementary particle somehow warps space on ultramicroscopic scales in its immediate neighbourhood. The uncertainty principle allows one to locate the particle spatially within a region bounded by a two-sphere S^2. Thus, by identifying S^2 to a point, one may characterise the particle by a closed, compact three-manifold M. By further requiring M to carry a tangent spinor structure, one equivalently requires M to be parallelisable. Thus the classification of homotopy classes of Lorentz metrics on closed, compact, parallelisable three-manifolds in some way lists the possible types of space–time structure on ultramicroscopic scales.

One can extend the above interpretation by noting that the uncertainty principle also limits localisability in time. If one visualises the space–time warp as some kind of four-dimensional knot travelling along a string, then one can only locate the knot within a region of space–time bounded by a three-sphere S^3. In this way, by identifying S^3 to a point, one obtains a compact, closed four-manifold X which characterises the particle. It also turns out, fairly naturally, that X ought to be parallelisable, although this is not necessary for the interpretation. If a region bounded by S^3 contains no particle, then the region must be diffeomorphic to D^4, the closed four-disc. Then, by identifying S^3 to a point, one obtains a manifold X diffeomorphic to S^4. In the three-dimensional case, by considering space-like two-spheres formerly occupied and presently occupied by particles, one naturally obtains a cobordism, through space–time, of S^3 to M. Such three-dimensional cobordisms are trivial in the sense that this places no topological restriction on M. For four-dimensions, one would expect that 'empty' space–time S^4 be deformable into a manifold X containing a particle. The deformation might be achieved via a cobordism through a five-dimensional hyperspace since (Whiston 1974) any space–time is cobordant in the unoriented sense. If X is oriented cobordant to S^4, the further imposition of internal Lorentzian structure and spinor structure would be sufficient to ensure (Whiston 1978) that X is parallelisable.

The calculation of homotopy classes of Lorentz metric tensor fields on arbitrary parallelisable four-manifolds undertaken in this paper shows that only a limited class of four-manifolds have topology suitable for the basically three-dimensional kink theory where a unique kink number homomorphism occurs. In nature, many types of conserved particle numbers exist—baryon number, lepton number, and so on. Such a family of kink numbers is naturally obtained if one considers more general four-manifolds. Indeed, these numbers are counted by the Betti number b_3. For compact four-manifolds, $b_3 \geqslant 1$ is a natural consequence of Lorentz structure and the group $H^4(X, Z_2)$ may yield an additional parity-like generator to the parity represented by the projection of a kink state into $H^1(X, Z_2)$. The latter group contains inrmation on the PT invariance of kink states. For example, if a kink state has a trivial projection, it represents a homotopy class of space–time orientable Lorentz metrics—the kink states Φ and $PT \cdot \Phi$ are globally distinguishable. States with a non-trivial projection behave in some way analogous to the π^0 meson where π^0 and $PT \cdot \pi^0$ are indistinguishable.

It is with the motivation of the above speculative links between elementary particle theory and space–time topology in mind, as well as the relativistic classification, that this paper considers the classification of homotopy classes of Lorentz metrics on both compact and non-compact parallelisable four-manifolds. It should be borne in mind that the discussion of compact four-manifolds is mainly of relevance to kink theory because of the well known fact that compact space–times have closed time-like curves and are therefore of little physical interest as global space–time models.

In § 2, deformation classes of Lorentz metrics on Minkowski space are considered. Because of complexities of orientation, there are three notions of equivalence invariant under homotopy: (a) Lorentz metrics with no preferred spatio-temporal orientation; (b) oriented Lorentz metrics where space and time orientations may be simultaneously reversed; (c) Lorentz metrics with preferred space and time orientations.

The homotopy classes of the first and second kind of metric are in 1–1 correspondence with RP^3, the three-dimensional projective plane, whilst the homotopy classes of bi-oriented metrics are in 1–1 correspondence with S^3. On a parallelisable manifold X, the homotopy classes of Lorentz metric tensor fields may be put into 1–1 correspondence with the homotopy sets of pointed maps $[X, RP^3]$ and $[X, S^3]$. In the first half of this paper, such sets are calculated for arbitrary manifolds of dimension less than five, but the physical interpretation is limited to parallelisable X. It is demonstrated that $[X, S^3]$ is an Abelian group for $\dim(X) \leqslant 5$ and that $[X, RP^3]$ is an Abelian group if $\dim(X) \leqslant 4$. Having established the commutativity of the groups, their extension structures, and hence a list of their generators and their combination into kink states, are investigated using elementary group extension theory. The first half of the paper closes with the calculation of $[X, RP^3]$ for several examples of compact four-manifolds.

The second half of this paper presents an analysis of the spin structure of kink states on four-manifolds, the word 'spin' having its usual meaning in kink theory. It is demonstrated that kink states which lift to S^3 and have odd generalised kink number have the fermion-like property of changing sign under 360° rotations about any spatial axis. The generalised kink number is defined by mapping into $H^3(X, Z)$ and the parity of the projection is defined by mapping into $H^3(X, Z_2)$. Note that if X is a compact parallelisable manifold of dimension three or four, the definition always makes sense because the relevant cohomology groups are non-trivial, although interpretative difficulties arise if torsion is present in the integral cohomology group.

Shastri *et al* obtained a similar result for the case when X is homotopy equivalent to a three-manifold of a special type—their type 2. Their result is extended to any three- or four-manifold for which the above definition of odd kink number makes sense, by a detailed analysis of the rotation which makes the role of the kink state explicit. It is interesting to note that the above result is consistent with the interpretation of the (free part of) $H^3(X, Z)$ as a set of conserved particle numbers. For example, the parity of an odd fermion state is independent of whether the fermions are leptons or baryons.

2. Deformation classes of Lorentz metrics on R^4

It is well known that the set of non-degenerate, symmetric, real, non-oriented bilinear forms of signature (p, q) on R^{p+q} is retractable onto the space $\mathrm{O}(p+q)/(\mathrm{O}(p)\times\mathrm{O}(q))$ of non-oriented orthogonal decompositions $R^{p+q} \cong R^p \oplus R^q$. The retraction is defined by noting that a bilinear form of signature (p, q) defines a decomposition of R^{p+q} into a sum of positive and negative definite eigenspaces. The inclusion map of $\mathrm{O}(p+q)/(\mathrm{O}(p)\times\mathrm{O}(q))$ into the space of forms sends a decomposition with associated projection maps P, Q into the quadratic form $(x, y) = \langle P(x), P(y)\rangle - \langle Q(x), Q(y)\rangle$ where P and Q $\langle\, ,\rangle$ denotes a positive definite quadratic form. The above inclusion map is a left homotopy inverse to the retraction and thus, from the homotopy theoretic viewpoint, any quadratic form of signature (p, q) is deformable into one of the type defined above. Suppose that one chooses an orientation of R^{p+q} consistent with semi-orientations on R^p and R^q. Then the corresponding space of orthogonal decompositions is replaced by $\mathrm{SO}(p+q)/(\mathrm{SO}(p)\times\mathrm{SO}(q))$, since in this case $\mathrm{SO}(p+q)$ acts transitively on the orthogonal decompositions with isotropy group $SO(p)\times SO(q)$. If one relaxes the requirement of orthogonality, one can alternatively start with the fact that $GL(p+q)$ acts transitively on the space of signature-(p, q) bilinear forms with isotropy group $O(p, q)$ and there is hence a bijection of the space of forms onto $GL(p+q)/O(p, q)$. The inclusion map of $O(p+q)$ into $GL(p+q)$ induces an inclusion map

$$(O(p+q), O(p+q)\cap O(p, q))\subset (GL(p+q), O(p, q)),$$

and because $O(p+q)\cap O(p, q) = O(p)\times O(q)$, the inclusion map mentioned above is obtained. The Lie group $O(p, q)$ has four connected components defined by the onto homomorphisms $\sigma_1, \sigma_2\colon O(p, q)\to Z_2$ where $\sigma_1\colon x\mapsto \mathrm{sgn}[\det(x)]$ and $\sigma_2\colon x\mapsto \mathrm{sgn}\{\det[\phi(x)]\}$ $\phi(x)$ being defined by $\phi(x)(R^p) = x\cdot(R^p)\cap R^p$. The subgroup $\mathrm{Ker}(\sigma_1)$ is obviously just $SL(p+q)\cap O(p, q) = SO(p, q)$ and $\mathrm{Ker}(\sigma_2)$, which will be denoted by $O_+(p, q)$, is the subgroup of $O(p, q)$ preserving the orientation on R^p. The subgroup $O_+(p, q)\cap SO(p, q) = SO_+(p, q)$ is the index four component of the identity and bi-preserves the semi-orientations on R^p and R^q. In short, there are the following deformation classes of oriented bilinear forms of signature (p, q) on R^{p+q}:

(1) non-oriented forms classified by

$$O(p+q)/(O(p)\times O(q))\subset GL(p+q)/O(p, q) = LO;$$

(2) consistently semi-oriented, oriented forms classified by

$$SO(p+q)/(SO(p+q)\cap SO(p, q))\subset SL(p+q)/SO(p, q) = LS;$$

(3) bi-oriented, oriented forms classified by

$$SO(p+q)/(SO(p)\times SO(q))\subset SL(p+q)/SO_+(p, q) = LS_+.$$

Note that $O(p, q)\cap SO(p+q) = (SO(p)\times SO(q))\cup(P\cdot SO(p)\times Q\cdot SO(q))$ where P and Q are diagonal orientation reversing maps in $O(p)$ and $O(q)$. That is, $x\in O(p, q)\cap SO(p+q)$ if x simultaneously either preserves or reverses the semi-orientations in R^p and R^q. The sets $O(p, q)\cap SO(p+q)$ and $SO(p)\times SO(q)\times Z_2(\alpha)$ are in bijective correspondence where α is the antipodal map on R^{p+q}. Specialising to $(p, q) = (1, 3)$, LO, LS and LS_+ are deformable into the subspaces $RP^3\cong O(4)/(O(1)\times O(3))$, $RP^3\cong SO(4)/O(3)$ and $S^3\cong SO(4)/SO(3)$. Note that $SO(3)\times Z_2(PT)$ is isomorphic to $O(3)\subset SO(4)$ via the injection $x\mapsto(s(x)\cdot x, s(x))$ where $s(x) = \mathrm{sgn}[\det(x)]$ for $x\in O(3)$.

3. Lorentz metric tensor fields

The Lorentz signature quadratic forms on R^4 may be split into homotopy classes as defined above with representatives generated by any one-plane in R^4. Given a four-manifold X, there is an analogous construction in each tangent space and a fibre bundle over X with fibre LO, LS or LS_+, and the bundle is naturally associated with the tangent frame bundle. Suppose that $\pi\colon E\to X$ denotes the RP^3 bundle. Then the set of homotopy classes of cross sections can be identified with the fibre $S(X) = \pi_{\#}^{-1}(|1_X|)\subset[X, E]$ where 1_X is the identity map of X. The calculation of the above set is a formidable task for arbitrary bundles, but for trivial bundles (e.g. the tangent bundle of a parallelisable manifold) it is easy to show that $S(X)$ is in 1–1 correspondence with either $[X, RP^3]$ or $[X, S^3]$.

If X is a parallelisable manifold, the homotopy classes of tangent Lorentz metrics are generated by unoriented lines $\theta \in RP^3$. The question then arises of assigning a global direction to the lines over the points of X, that is, given lines $f(x) \subset T_x$, and that $f(x)$ represents points z and $PT \cdot z$ in S_x^3, can one globally choose say, z, i.e. does f lift to S^3? This question can be answered using elementary covering space theory. A standard result states that given a cover $p: \tilde{Y} \to Y$, a map $f: X \to Y$ lifts to $\tilde{Y}$ if and only if $\pi_1(f)(\pi_1(X)) \subset \pi_1(p)(\pi_1(Y))$. Hence if Y is simply connected, $|f|$ lifts to $\tilde{Y}$ if and only if $\pi_1(f) = 0$. Consider the Z_2 bundle $\sigma: S^3 \to RP^3 = S^3/Z_2$. Then we have established that the following sequence of sets and mappings is exact:

$$[X, S^3] \xrightarrow[\sigma_\#]{} [X, RP^3] \xrightarrow[W]{} \mathrm{Hom}(\pi_1(X), Z_2)$$

where $W: |f| \mapsto \pi_1(f)$. The spaces S^3 and RP^3 are Lie groups and the projection $\sigma: S^3 \to RP^3$ is isomorphic to Spin(3) $\to$ SO(3), which is a group homomorphism. Hence $[X, S^3]$ and $[X, RP^3]$ are groups under the multiplication induced by the point-wise multiplication of maps and $\sigma_\#$ is a group homomorphism. The map W is also a group homomorphism, because if $m: RP^3 \times RP^3 \to RP^3$ is the multiplication map, the induced homotopy homomorphism $\pi_1(m)$ coincides with the usual addition in $\pi_1(RP^3)$. The above exact sequence can be revamped slightly by replacing $\mathrm{Hom}(\pi_1(X), Z_2)$ by the isomorphic group $H^1(X, Z_2)$. The resulting homomorphism, again denoted by W, maps a homotopy class of maps $|f|$ into the cohomology class $f^*(W_1)$ where W_1 is the Stiefel–Whitney class of the Z_2 bundle $\sigma: S^3 \to RP^3$.

In order to obtain more information on the above sequence, one has to dig a little deeper. Regarding $S^3 \to RP^3$ as Spin(3) $\to$ SO(3) and $H^1(X, Z_2)$ as $[X, K(Z_2, 1)]$, the above sequence is part of the Puppe sequence for the fibration (Avis and Isham 1980)

$$\begin{array}{ccc} BZ_2 = K(Z_2, 1) & \xrightarrow{Bi} & B\mathrm{Spin}(3) \\ & & \downarrow{\scriptstyle B\sigma} \\ & & B\mathrm{SO}(3) \xrightarrow[W_2]{} K(Z_2, 2) \end{array}$$

where $B\sigma$ and Bi are induced by the extension $Z_2 \overset{i}{\rightarrowtail} \mathrm{Spin}(3) \overset{\sigma}{\twoheadrightarrow} \mathrm{SO}(3)$ and W_2 is the second universal Stiefel–Whitney class. After a little rearranging, the Puppe sequence becomes

$$\begin{array}{l} [X, Z_2] \xrightarrow[i_\#]{} [X, \mathrm{Spin}(3)] \xrightarrow[\sigma_\#]{} [X, \mathrm{SO}(3)] \xrightarrow[W]{} H^1(X, Z_2) \\ \quad \xrightarrow[Bi_\#]{} [X, B\mathrm{Spin}(3)] \xrightarrow[B\sigma_\#]{} [X, B\mathrm{SO}(3)] \xrightarrow[W_2]{} H^2(X, Z_2). \end{array}$$

Firstly, $[X, Z_2] = 0$ for a path connected space, which implies that $\sigma_\#$ is injective because it is a group homomorphism. According to Avis and Isham (1980), $B\sigma_\#$ is an injective map for four-manifolds, i.e. a Spin(3) bundle on a four-manifold is uniquely determined by the SO(3) bundle that it covers, each being determined by the Pontryagin class in $H^4(X, Z)$. This implies that W is an epimorphism. The method of classification is now clear. If $f^*(W) = 0$ for $|f| \in [X, RP^3]$, $|f|$ lifts uniquely to $[X, S^3]$, S^3 being one stage simpler topologically than RP^3. Of course, if $|f| \notin \mathrm{Im}(\sigma_\#)$, $|f|$ is only specified by the characteristic class $f^*(W_1)$ up to an element of $[X, S^3]$. One would like to be able to define a splitting homomorphism $[X, RP^3] \to [X, S^3]$ which would fully determine $|f|$ in terms of invariants of $[X, S^3]$ as well as $f^*(W_1)$. That is, one needs to know what type of group extension $[X, RP^3]$ is and whether or not it is Abelian. Having established that $[X, RP^3]$ is a group extension of $[X, S^3]$ by $H^1(X, Z_2)$ for any connected four-manifold,

$$[X, S^3] \underset{\sigma_{\#}}{\rightarrowtail} [X, RP^3] \underset{W}{\twoheadrightarrow} H^1(X, Z_2),$$

the next stage is to calculate the group $[X, S^3]$.

4. The group $[X, S^3]$

Because S^3 is two-connected, it is a relatively simple matter to obtain a Postnikov resolution (Thomas 1966, Avis and Isham 1979, Isham 1981), i.e. a resolution of the constant fibration $S^3 \xrightarrow{\lambda} 0$ up to stage four. We need to know the homotopy groups $\pi_q(S^3)$ for $0 \leqslant q \leqslant 4$:- $\pi_q(S^3) = 0$ for $q < 3$, $\pi_3(S^3) \cong Z$ and $\pi_4(S^3) \cong Z_2$. The construction starts at level three:

$$\begin{tikzcd}
S^3 \arrow[r, equal] \arrow[d, "v"'] & S^3 \arrow[d, "q_3"] \arrow[dd, bend left=60, "\lambda"] & \\
K(Z, 3) \arrow[r, "i"'] & E_3 \arrow[d, "p_3"] & \\
 & 0 \arrow[r, "\theta"'] & K(Z, 4).
\end{tikzcd}$$

Because $\theta \circ \lambda$ is trivially null homotopic for any θ, λ lifts to E_3, which is the pull-back to 0 of the path fibration over $K(Z, 4)$. Thus $E_3 \cong K(Z, 3)$ and i can be taken as the identity map. If we choose $v = q_3 = \langle S^3 \rangle$, the generator of $H^3(S^3, Z)$, q_3 is obviously a three-equivalence. The next stage of the resolution, level four, involves the lifting of q_3 to a four-equivalence q_4:

$$\begin{tikzcd}
F_{q_3} \arrow[r, "j"'] \arrow[d, "v"'] & S^3 \arrow[d, "q_4"'] \arrow[dd, bend left=60, "q_3"] & \\
K(Z_2, 4) \arrow[r, "i"'] & E_4 \arrow[d, "p_4"'] & \\
 & K(Z, 3) \arrow[r, "\theta_2"'] & K(Z_2, 5).
\end{tikzcd}$$

F_{q_3} is the homotopy fibre of q_3, i and j are inclusion maps and v is the map on F_{q_3} induced by q_4. The latter map is a lifting of q_3 to E_4 which is the pull-back to $K(Z, 3)$ of the path fibration over $K(Z_2, 5)$ along a cohomology operation $\theta_2 \in H^5(K(Z, 3), Z_2)$. Such a lifting will always exist for any θ_2 because $\theta_2 \circ q_3 \simeq 0$ follows from $H^5(S^3, Z) = 0$. In order to obtain a four-equivalence we have to choose θ_2 such that $\pi_4(v)$ is an isomorphism, and to ascertain which θ_2 to use, we examine the Serre exact sequence of the fibration q_3, since a standard result asserts that the set of appropriate θ_2 lies in the image of the transgression relation T from fibre into base cohomology

$$H^4(K(Z, 3), Z_2) \xrightarrow[q_3^*]{} H^4(S^3, Z_2) \xrightarrow[j^*]{} H^4(F_{q_3}, Z_2) \xrightarrow[T]{} H^5(K(Z, 3), Z_2).$$

Note that the groups $H^4(S^3, Z_2)$ and $H^5(S^3, Z_2)$ are trivial and therefore T is an isomorphism. In this case, any non-trivial θ_2 induces a q_4 which is a four-equivalence. According to a result of Serre (1972), $H^5(K(Z, 3), Z_2)$ is generated by the cohomology operation $\mathrm{Sq}^2 \circ \pi_2$ where Sq^2 is the usual mod(2) Steenrod square and π_2 is the coefficient homomorphism induced by the extension $Z \overset{2}{\rightarrowtail} Z \overset{\pi_2}{\twoheadrightarrow} Z_2$. We note that $\Omega(\mathrm{Sq}^2 \circ \pi_2)$ coincides with the generator $\mathrm{Sq}^2 \circ \pi_2$ of $H^4(K(Z, 2), Z)$ which is the mapping $x \mapsto |x|_2 \cup |x|_2$ on two-dimensional cohomology classes x. Recall that the class $|q_3| \in H^3(S^3, Z)$ is the generator. Thus, because q_4 is a four-equivalence and $q_4^*(|p_4|) = |q_3|$, we may identify $|p_4|$ with the generator $\langle S^3 \rangle$ of $H^3(E_4, Z)$ when identifying E_4 with S^3 for four-manifold calculations. Having detailed the main mappings in the resolution, we consider the Puppe sequence of the fibration p_4:

$$H^2(X, Z) \xrightarrow[\mathrm{Sq}^2 \circ \pi_2]{} H^4(X, Z_2) \xrightarrow[i_\#]{} [X, S^3] \xrightarrow[p_{4\#}]{} H^3(X, Z) \xrightarrow[\mathrm{Sq}^2 \circ \pi_2]{} H^5(X, Z_2).$$

For four-manifolds, $H^5(X, Z_2) = 0$ so that $p_{4\#}: |f| \mapsto |p_4 \circ f| \equiv f^*(|p_4|) = f^*\langle S^3 \rangle$ is a surjective function. We have therefore proved the following result, which is a special case of the Steenrod classification theorem.

Theorem 1. Let X be a four-manifold. Then the group $[X, S^3]$ is part of the following exact sequence of groups and homomorphisms

$$H^2(X, Z) \xrightarrow[\mathrm{Sq}^2 \circ \pi_2]{} H^4(X, Z_2) \xrightarrow[i_\#]{} [X, S^3] \xrightarrow[p_{4\#}]{} H^3(X, Z)$$

where $p_{4\#}: |f| \mapsto f^*\langle S^3 \rangle$ and $\mathrm{Sq}^2 \circ \pi_2: x \mapsto |x|_2 \cup |x|_2$.

It remains to demonstrate that $p_{4\#}$ and $i_\#$ are homomorphisms. The proof that $p_{4\#}$ is a homomorphism is to be found in Shastri *et al* (1980) whilst $i_\#$ is non-trivial only if X is compact. For this case, $\mathrm{Im}(i_\#) = \mathrm{Ker}(p_{4\#})$ is a subgroup (because $p_{4\#}$ is a homomorphism) of order two in $[X, S^3]$. This implies that its non-trivial element must be of order two and hence that $i_\#$ is a homomorphism.

We now discuss the implications of theorem 1 for arbitrary four-manifolds.

(1) *Non-compact four-manifolds*

As we noted above, $H^4(X, Z_2) = 0$ if X is non-compact and it therefore follows that $p_{4\#}$ is an isomorphism of $[X, S^3]$ onto $H^3(X, Z)$.

(2) *Compact four-manifolds*

If X is compact, the structure of $[X, S^3]$ is a little more complicated, for we always have $H^4(X, Z_2) \cong Z_2$. Moreover, if X is parallelisable, or just a space–time, $\chi(X) = 0$ implies that b_3, the third Betti number of X, is non-zero. The structure of $H^3(X, Z)$ is complicated by the possible presence of torsion, but this torsion is easy to spot, being generated by torsion in $\pi_1(X)$. We are interested in the homomorphism $i_\#$ and hence $\mathrm{Sq}^2 \circ \pi_2$. If X is a four-manifold, there is a characteristic class $V \in H^2(X, Z_2)$ called the Wu class of X such that for all $x \in H^2(X, Z_2)$, $\mathrm{Sq}^2(x) = x \cup x = V \cup x$. It can be shown that for manifolds of dimension divisible by four, where a similar construction can be performed, that V is a function of Stiefel–Whitney classes generated by Wu's formula (Spanier 1966) which reduces to $V = W_2$ in dimension four. Therefore, given $x \in H^2(X, Z)$, $\mathrm{Sq}^2 \circ \pi_2(x) = W_2 \cup |x|_2$. There are the following cases to consider:

(a) $Sq^2 \circ \pi_2 = 0$. In this case, $i_\#$ is a monomorphism and $[X, S^3]$ is a group extension of $H^4(X, Z_2)$ by $H^3(X, Z)$. Note that $\mathrm{Sq}^2 \circ \pi_2 = 0$ in the following cases:

(i) $\pi_2 = 0$ or equivalently, all elements of $H^2(X, Z)$ are divisible by two;

(ii) $\mathrm{Sq}^2=0$ or $W_2=0$: X is a spin manifold (true if X is parallelisable);

(iii) $\mathrm{Im}(\pi_2)\subset \mathrm{Ker}(\mathrm{Sq}^2)$;

(b) $Sq^2\circ\pi_2\neq 0$. If $\mathrm{Sq}^2\circ\pi_2\neq 0$, $i_{\#}$ is the trivial homomorphism and, as for the case of non-compact manifolds, $[X, S^3]$ is isomorphic to $H^3(X, Z)$.

The above calculations have linked the groups $[X, RP^3]$ and $[X, S^3]$ and have established extension structures in terms of cohomology groups. Because the theory of group extensions is much simpler for Abelian extensions, the following analyses are directed towards establishing commutativity. The following simple argument based on one presented in Shastri *et al* (1980), connects two-torsion in $[X, S^3]$ with commutativity in $[X, RP^3]$.

Proposition 1. Let X be any space such that $[X, S^3]$ is Abelian. Then if $[X, S^3]$ has no two-torsion, $[X, RP^3]$ is Abelian.

Proof. Consider the map $q: RP^3\to S^3$ defined by $q\circ\sigma=\mathbf{2}$ where $\mathbf{2}: S^3\to S^3$ is the squaring map $x\mapsto x^2$. $q_{\#}$ is a group homomorphism on $\mathrm{Im}(\sigma_{\#})$ if $[X, S^3]$ is Abelian because in that case, $\mathbf{2}_{\#}$ is an endomorphism. Let $aba^{-1}b^{-1}$ be a commutator in $[X, RP^3]$. Then the commutator lies in $\mathrm{Ker}(W)$ because $H^1(X, Z_2)$ is Abelian, and hence there is an element c of $[X, S^3]$ such that $aba^{-1}b^{-1}=\sigma_{\#}(c)$. But the commutator also lies in $\mathrm{Ker}(q_{\#})$ because $[X, S^3]$ is Abelian, and hence $q_{\#}\circ\sigma_{\#}(c)=c^2=0$. Therefore if $[X, S^3]$ has no two-torsion, $c=0$ which implies that $aba^{-1}b^{-1}=0$.

The above result only gives partial information on commutativity for a broad class of spaces, whilst the two following theorems state that $[X, S^3]$ is always Abelian for manifolds X with $\dim(X)\leqslant 5$ and that $[X, RP^3]$ is Abelian for manifolds X with $\dim(X)\leqslant 4$. The result on $[X, S^3]$ is remarkably direct, whilst the result on $[X, RP^3]$ involves a Postnikov resolution of a lifting of the commutator map on RP^3.

Theorem 2. If X is a manifold of dimension $\leqslant 5$, $[X, S^3]$ is an Abelian group.

Proof. Let $[\,,\,]: S^3\times S^3\to S^3$ be the commutator map on S^3: $[\,,\,]:(x, y)\mapsto xyx^{-1}y^{-1}$. Then the induced map $[\,,\,]_{\#}:[X, S^3\times S^3]\cong[X, S^3]\times[X, S^3]\to[X, S^3]$ defines the commutator on $[X, S^3]$. For, suppose that $|f|$, $|g|\in[X, S^3]$, then $[|f|, |g|]=|f|\,|g|\,|f|^{-1}|g|^{-1}=|fgf^{-1}g^{-1}|=|[\,,\,]\circ(f\times g)\circ\Delta|=[\,,\,]_{\#}(|(f, g)|)$ where (f, g) denotes the map $(f\times g)\circ\Delta$. Consider the following sequence of spaces and maps:

$$S^3\vee S^3\xrightarrow{J}S^3\times S^3\xrightarrow{P}S^3\wedge S^3\cong S^6.$$

Given a space Y and a map $q: S^3\times S^3\to Y$ such that $q\circ J=0$ on the coordinate subspace $S^3\vee S^3$, there is a continuous induced map q': $S^3\wedge S^3\to Y$ such that $q'\circ P=q$. It is easy to establish that $[\,,\,]\circ J=0$ (all elements commute with the identity) and hence there is a factorisation $[\,,\,]=\psi\circ P$ where $\psi: S^6\to S^3$. One therefore obtains a factorisation of the induced commutator map:

$$\begin{tikzcd}
{[X, S^3\times S^3]} \arrow[rr, "P_{\#}"] \arrow[dr, "{[,]_{\#}}"'] & & {[X, S^6]\cong H^6(X, Z)} \arrow[dl, "\psi_{\#}"] \\
& {[X, S^3]} &
\end{tikzcd}$$

It is easy to verify that S^6 is six-equivalent to $K(Z, 6)$, such an equivalence being realised by the fundamental class $\langle S^6\rangle$. Therefore $[X, S^6]$ may be replaced by $H^6(X, Z) = 0$ for $\dim(X) \leq 5$. Therefore $P_\#$ and hence $[\,,\,]_\#$ are trivial maps.

The following string of lemmas leads up to the main theorem of this part of this paper—that the group $[X, RP^3]$ is Abelian for manifolds X with $\dim(X) \leq 4$. Of course, the latter result will also imply the corresponding result about $[X, S^3]$. Theorem 2, which holds for $\dim(X) = 5$, is needed in the analysis of the spin properties of kink states when information on $[S^1 \wedge X, S^3]$ for a four-manifold X is needed. The calculation involves a level-5 Postnikov resolution of S^3 and will form the subject of the second half of this paper.

Lemma 1. The commutator map $[\,,\,]: RP^3 \times RP^3 \to RP^3$ has a homotopy lift to S^3.

Proof. The following sequence is exact for *any* space Y:

$$[Y, S^3] \xrightarrow{\sigma_\#} [Y, RP^3] \xrightarrow{W} H^1(Y, Z_2)$$

where $W(|f|) = f^*(W_1)$ for W_1 the Stiefel–Whitney class of $S^3 \to RP^3$. Let $Y = RP^3 \times RP^3$ and $f = [\,,\,]$. Then $[\,,\,]$ lifts to S^3 if and only if $[\,,\,]^*(W_1) = 0$. But this follows immediately from the fact that $[\,,\,] \circ i_1$ and $[\,,\,] \circ i_2$ are constant maps for i_1 and i_2 the canonical injections of the (pointed) product space: all elements commute with the identity. Hence there exists a unique homotopy class of maps $|\gamma|$ such that $|[\,,\,]| = \sigma_\#(|\gamma|)$.

The map $\gamma_\#: [X, RP^3 \times RP^3] \to [X, S^3]$ determines the commutator on $[X, RP^3]$ because $\sigma_\#$ is a monomorphism—$[|f|, |g|] = 0$ if and only if $\gamma_\#(|(f, g)|) = 0$. We next have to analyse the map $\gamma_\#$ which is induced by γ on homotopy classes of maps with domain a four-manifold and, since S^3 is four-equivalent to E_4, it will be sufficient to discuss the associated map $q_4 \circ \gamma$. In turn, the map $q_4 \circ \gamma$ is specified by $p_4 \circ (q_4 \circ \gamma) = q_3 \circ \gamma$, that is, by the class $\gamma^*(|q_3|) = \gamma^*\langle S^3\rangle$ of $H^3(RP^3 \times RP^3, Z)$. Consider the following lemma.

Lemma 2. $\gamma^*\langle S^3\rangle$ is a two-torsion element and there exists a class C of Z_2 such that $\gamma^*\langle S^3\rangle = \beta_2(C \cdot W_1 \otimes W_1)$ where β_2 is the Bockstein homomorphism $H^2(RP^3 \times RP^3, Z_2) \to H^3(RP^3 \times RP^3, Z)$. Alternatively, $\pi_2(\gamma^*\langle S^3\rangle) = \mathrm{Sq}^1(C \cdot W_1 \otimes W_1)$.

Proof. According to the Kunneth theorem, $H^3(RP^3 \times RP^3, Z)$ decomposes as

$$\bigoplus_{p+q=3} H^p(RP^3) \otimes H^q(RP^3) \underset{\mu}{\rightarrowtail} H^3(RP^3 \times RP^3) \twoheadrightarrow \bigoplus_{p+q=4} \mathrm{Tor}(H^p(RP^3), H^q(RP^3))$$

where $H^k(RP^3)$ denotes integral cohomology. The image of a tensor product class $x \otimes y$ under μ is usually written as the cohomology cross product $x \times y$, but we shall usually identify $x \times y$ with $x \otimes y$. The exact sequence is split so that $H^3(RP^3 \times RP^3)$ has generators $x_3 \otimes 1$, $1 \otimes x_3$ and t_2 where t_2 is a two-torsion element arising from the summand $\mathrm{Tor}(H^2(RP^3), H^2(RP^3)) \cong \mathrm{Tor}(Z_2, Z_2) \cong Z_2$ and x_3 denotes the generator of $H^3(RP^3)$. Given these generators, there exist integers a, b, c such that

$$\gamma^*\langle S^3\rangle = a \cdot (x_3 \otimes 1) + b \cdot (1 \otimes x_3) + c \cdot t_2.$$

We claim that $a = b = 0$. To see this, recall that $[\,,\,] \circ i_1 = [\,,\,] \circ i_2 = 0$ for $i_1 = (1 \times 0) \circ \Delta$

and $i_2 = (0 \times 1) \circ \Delta$. This implies that $\gamma \circ i_1 \simeq 0$ and $\gamma \circ i_2 \simeq 0$ because $\sigma_{\#}$ is a monomorphism and hence that $i_1^* \circ \gamma^*\langle S^3 \rangle = 0$ and $i_2^* \circ \gamma^*\langle S^3 \rangle = 0$. But

$$i_1^* \circ \gamma^*\langle S^3 \rangle = \Delta^*(a \cdot (x_3 \otimes 0^*(1))) + \Delta^*(b \cdot (1 \otimes 0^*(x_3))) + i_1^*(t_2).$$

It then follows that $a = 0$ because $i_1^*(t_2) = 0$, being a two-torsion element in a free Abelian group. Similarly, $b = 0$ and hence $\gamma^*\langle S^3 \rangle$ is a two-torsion element, i.e. $\gamma^*\langle S^3 \rangle \in \mathrm{Ker}(2_*) = \mathrm{Im}(\beta_2)$ in the Bockstein coefficient sequence associated with $Z \overset{2}{\rightarrowtail} Z \overset{\pi_2}{\twoheadrightarrow} Z_2$. The group $H^2(RP^3 \times RP^3, Z_2)$ has generators $|x_2|_2 \otimes 1$, $1 \otimes |x_2|_2$ and $W_1 \otimes W_1$ where x_2 generates $H^2(RP^3)$, and hence $\mathrm{Im}(\beta_2)$ is generated by $\beta_2(W_1 \otimes W_1)$ because $\mathrm{Ker}(\beta_2) = \mathrm{Im}(\pi_{2*})$ which is generated by $|x_2|_2 \otimes 1$ and $1 \otimes |x_2|_2$. It then follows immediately that $\gamma^*\langle S^3 \rangle = \beta_2(C \cdot (W_1 \otimes W_1))$ for some class C of Z_2.

We therefore regain an earlier result: if $i_{\#} = 0$ and $H^3(X, Z)$ has no two-torsion, $[X, RP^3]$ is an Abelian group. For then $p_{4\#}$ is an isomorphism and hence $[|f|, |g|] = 0$ if and only if $(f, g)^* \circ \gamma^*\langle S^3 \rangle = (f, g)^*(c \cdot t_2) = 0$ which is certainly true if $H^3(X, Z)$ has no two-torsion. It also follows that because β_2 is functorial with respect to induced cohomology homomorphisms, the commutator $[|f|, |g|]$ is specified by $\beta_2(C \cdot (f^*(W_1) \cup g^*(W_1)))$. For example, if $f^*(W_1) = g^*(W_1)$, $|f|$ commutes with $|g|$ because $f^*(W_1) \cup g^*(W_1) = f^*(W_1)^2 = f^*(|x_2|_2) = f^*(x_2)_2$ which is in $\mathrm{Ker}(\beta_2)$. In order to determine the universal obstruction C which will yield results for all manifolds X of dimension less than four, we consider the case of $X = S^1 \times RP^3$ in detail, the first task being to establish that $[S^1 \times RP^3, RP^3]$ is an Abelian group independent of C.

Example: the group $[S^1 \times RP^3, RP^3]$
Let $i_1, i_3 : S^1, RP^3 \to S^1 \times RP^3$ and $p_1, p_3 : S^1 \times RP^3 \to S^1, RP^3$ be the canonical inclusion and projection maps. It is then straightforward to establish that the following mesh of groups and homomorphisms commutes and is exact along any vertical or horizontal three-map segment which starts on part of a group extension:

$$\begin{tikzcd}
{[RP^3, S^3]} \arrow[r, tail, "\sigma_{\#}"'] \arrow[d, dotted, shift right, "\hat{p}_3^{\#}"'] & {[RP^3, RP^3]} \arrow[r, two heads, "W"'] \arrow[d, dotted, shift right, "p_3^{\#}"'] & H^1(RP^3, Z_2) \arrow[d, dotted, shift right, "p_3^*"'] \\
{[S^1 \times RP^3, S^3]} \arrow[r, tail, "\sigma_{\#}"'] \arrow[u, shift right, "\hat{i}_3^{\#}"'] & {[S \times RP^3, RP^3]} \arrow[r, two heads, "W"'] \arrow[u, shift right, "i_3^{\#}"'] & H^1(S^1 \times RP^3, Z_2) \arrow[u, shift right, "i_3^*"'] \arrow[d, shift left, "i_1^*"] \\
\mathrm{Ker}(\hat{i}_3^{\#}) \arrow[r, tail, "\sigma_{\#}"'] \arrow[u, "\mathrm{inc}"] & \mathrm{Ker}(i_3^{\#}) \arrow[r, two heads, "W"'] \arrow[u, "\mathrm{inc}"'] & H^1(S^1, Z_2) \arrow[u, shift left, "p_1^*"] \arrow[d, equal] \\
 & {[S^1, RP^3]} \arrow[r, tail, two heads, "W"'] \arrow[u, "p_1^{\#}"'] & H^1(S^1, Z_2).
\end{tikzcd}$$

This exhibits $[S^1 \times RP^3, RP^3]$ as a group extension of $\mathrm{Ker}(i_3^{\#})$ by $[RP^3, RP^3]$ and $\mathrm{Ker}(i_3^{\#})$ as a group extension of $\mathrm{Ker}(\hat{i}_3^{\#})$ by $H^1(S^1, Z_2)$. The superscript '^' refers to homomorphisms associated with S^3 and the dotted homomorphisms are cross sections which split the extensions on the right. In turn, the group $\mathrm{Ker}(\hat{i}_3^{\#})$ fits into the following morphism of group extensions:

$$\begin{tikzcd}
H^4(S^1 \times RP^3, Z_2) \arrow[r, tail, "i_{\#}"] \arrow[d, equal] & \mathrm{Ker}(\hat{i}_3^{\#}) \arrow[r, two heads, "p_{4\#}"] \arrow[d, "\mathrm{inc}"] & H^2(RP^3, Z) \otimes H^1(S^1, Z) \arrow[d, "\mathrm{inc}"] \\
H^4(S^1 \times RP^3, Z_2) \arrow[r, tail, "i_{\#}"'] & {[S^1 \times RP^3, S^3]} \arrow[r, "p_{4\#}"'] & H^3(S^1 \times RP^3, Z).
\end{tikzcd}$$

We note that if $|\theta| \in H^4(S^1 \times RP^3, Z_2)$, $\hat{i}_3^{\#}(|i \circ \theta|) = 0$ because $i_3^*(|\theta|) \in H^4(RP^3, Z_2) = 0$, i.e. $\mathrm{Im}(i_{\#}) \subset \mathrm{Ker}(\hat{i}_3^{\#})$. Therefore $\mathrm{Ker}(\hat{i}_3^{\#})$ is a group extension of Z_2 by Z_2 and is consequently Abelian because the only groups of order four, Z_4 and $Z_2 \oplus Z_2$, are Abelian. In fact, $\mathrm{Ker}(\hat{i}_3^{\#})$ is split.

Lemma 3. $\mathrm{Ker}(\hat{i}_3^{\#})$ is isomorphic to $Z_2 \oplus Z_2$.

Proof. If $|g| \in \mathrm{Ker}(\hat{i}_3^{\#})$, $g \circ i_3 \simeq 0$ and automatically, $g \circ i_1 \simeq 0$ because $\pi_1(S^3) = 0$. Hence if $J: S^1 \vee RP^3 \subset S^1 \times RP^3$ is the inclusion of the coordinate axes and $P: S^1 \times RP^3 \to S^1 \wedge RP^3 \equiv \Sigma RP^3$ is the projection onto the suspension, $|g| = P_{\#}(|\rho|)$ for some homotopy class $|\rho| \in [RP^3, S^3]$, i.e. $P^{\#}: [\Sigma RP^3, S^3] \to \mathrm{Ker}(\hat{i}_3^{\#})$ is surjective. Consider the following commutative diagram:

$$\begin{tikzcd}
H^4(S^1 \times RP^3, Z_2) \arrow[r, tail, "i_{\#}"] & \mathrm{Ker}(\hat{i}_3^{\#}) \arrow[r, two heads, "p_{4\#}"] & H^2(RP^3, Z) \otimes H^1(S^1, Z) \\
H^4(\Sigma RP^3, Z_2) \arrow[u, "P^*"] \arrow[r, tail, "i_{\#}"'] & {[\Sigma RP^3, S^3]} \arrow[u, "P_{\#}"] \arrow[r, two heads, "p_{4\#}"'] & H^3(\Sigma RP^3, Z) \arrow[u, "P^*"] \\
H^3(RP^3, Z_2) \arrow[u, "\Sigma", "\cong"'] & & H^2(RP^3, Z). \arrow[u, "\Sigma", "\cong"']
\end{tikzcd}$$

It is easy to establish that the diagram commutes and it then follows from the five-lemma that $P^{\#}$ is an isomorphism. According to Shastri *et al* (1980) $[\Sigma RP^3, S^3]$ is split. Therefore $\mathrm{Ker}(\hat{i}_3^{\#})$ is also split.

Lemma 4. $\mathrm{Ker}(i_3^{\#})$ is isomorphic to $\mathrm{Ker}(\hat{i}_3^{\#}) \oplus Z_2$.

Proof. Referring to the main diagram decomposing $[S^1 \times RP^3, RP^3]$ into group extensions, $\mathrm{Ker}(i_3^{\#})$, which is not necessarily Abelian, is a split extension of $\mathrm{Ker}(\hat{i}_3^{\#})$ by $H^1(S^1, Z_2)$. Hence $\mathrm{Ker}(i_3^{\#})$ is a group of order eight by Lagrange's theorem. There are only five groups of order eight, of which only two are non-Abelian; these are: (1) Z_8 (2) $Z_4 \oplus Z_2$ (3) Z_2^3 (4) D_4 and (5) D_2^* where D_4 is the dihedral group of the square and D_2^* is the dicyclic group with two elements of order four. The group D_4 is a split extension of Z_4 by Z_2 and D_2^* is a non-split extension of Z_4 by Z_2. It therefore follows that $\mathrm{Ker}(i_3^{\#})$ is Abelian (having a split kernel) and since it is a split extension, it must be isomorphic to Z_2^3.

Lemma 5. $[S^1 \times RP^3, RP^3]$ is Abelian (and isomorphic to $Z_2^3 \oplus Z$).

Proof. According to Shastri *et al* (1980) $[RP^3, RP^3] \cong Z$. Therefore, $[S^1 \times RP^3, RP^3]$ is a semi-direct product of Z_2^3 by Z and, as such, is determined by a homomorphism of $\mathrm{Hom}(Z, \mathrm{Aut}(Z_2^3))$, itself determined by the automorphism $A(1)$ defined by the inner automorphism induced by $|p_3|$. It is easy to see that the latter automorphism is trivial on $\mathrm{Ker}(\hat{i}_3^{\#})$ and hence that $A(1)$ is determined by its action on $p_1^{\#}(|\gamma|)$, where $|\gamma|$ is the homotopy class of the 360° rotation about an arbitrary axis $\boldsymbol{r}$ in R^3 representing a generator of $\pi_1(RP^3)$:

$$|p_3| p_1^{\#}(|\gamma|) |p_3|^{-1} = |f| \qquad \text{for } f(x, y) = y\gamma(x)y^{-1}.$$

We claim that $f \simeq \gamma \circ p_1$. To see this, note that the map ϕ from RP^3 into the space of maps from S^1 into RP^3 defined by: $\phi(y)(x)=f(x, y)$ is continuous, each $\phi(y)$ being a rotation about an axis $y(r)$ and hence homotopic to $\phi(e)=\gamma$. Therefore A is trivial and $[S^1\times RP^3, RP^3]$ is Abelian.

Lemma 6. The universal obstruction C of lemma 2 is trivial.

Proof. The homomorphism $\mathrm{Sq}^2\circ\pi_2$ is trivial because the generator $x_2\otimes 1$ of the group $H^2(S^1\times RP^3, Z)$ is mapped into $|x_2|_2^2\otimes 1=W_1^4\otimes 1=0$ because $H^4(RP^3, Z_2)$ is trivial. The homomorphism β_2 is defined on the generators of $H^2(S^1\times RP^3, Z_2)$ by $|x_2|_2\otimes 1\mapsto 0$ (because $|x_2|_2\otimes 1\in \mathrm{Im}(\pi_{2*})$) and $\beta_2(W_1\otimes|y_1|_2)=y_1\otimes x_2$. According to lemma 5, $[S^1\times RP^3, RP^3]$ is an Abelian group. Thus for any homotopy classes $|f|$, $|g|$, $[|f|, |g|]=\sigma_{\#}\circ\gamma_{\#}(|(f, g)|)=0$ which implies that $\gamma_{\#}(|(f, g)|)=0$ because $\sigma_{\#}$ is a monomorphism. But then $p_4\circ\gamma_{\#}(|(f, g)|)=\beta_2(C\cdot(f^*(W_1)\cup g^*(W_1)))=0$. Hence if there exist classes $|f|$, $|g|$ such that $f^*(W_1)\cup g^*(W_1)\notin \mathrm{Ker}(\beta_2)$, it must follow that $C=0$. Let $f: S^1\times RP^3\to RP^3$ be the projection p_3 and let $g: S^1\times RP^3\to RP^3$ be $k\circ p_1$ where k represents the generator of $\pi_1(RP^3)$. Then $f^*(W_1)=(1\otimes W_1)$, $g^*(W_1)=|y_1|_2\otimes 1$ and $f^*(W_1)\cup g^*(W_1)=|y_1|_2\otimes W_1\notin \mathrm{Ker}(\beta_2)$. Therefore $C=0$.

Corollary. The map $q_4\circ\gamma$ lifts to $K(Z_2, 4)$. That is, there exists a universal obstruction $|\theta|\in H^4(RP^3\times RP^3, Z_2)$ such that $q_4\circ\gamma\simeq i\circ\theta$.

The above obstruction θ determines the commutator on $[X, RP^3]$ for four-manifolds X in that $[|f|, |g|]$ is trivial if and only if $i_{\#}((f, g)^*(|\theta|))=0$. Note that this already establishes that $[X, RP^3]$ is Abelian for those manifolds with $i_{\#}=0$, and consequently we only need to consider manifolds with non-trivial $i_{\#}$. Using the Kunneth theorem, it follows that $H^4(RP^3\times RP^3, Z_2)$ has Z_2 basis vectors $W_1^2\otimes W_1^2$, $W_1^3\otimes W_1$ and $W_1\otimes W_1^3$. There must thus exist a vector (A, B, C) of Z_2^3 such that

$$|\theta|=A\cdot(W_1^2\otimes W_1^2)+B\cdot(W_1^3\otimes W_1)+C\cdot(W_1\otimes W_1^3).$$

The following lemma establishes that $(A, B, C)=0$.

Lemma 7. $(A, B, C)=0$.

Proof. (1) Let T be the transposition diffeomorphism of $RP^3\times RP^3$. It then follows from the property $[\,,\,]\circ T=I\circ[\,,\,]$ where $I: x\mapsto x^{-1}$ that $T^*(|\theta|)=|\theta|$. But then it is easy to show that this implies that $B=C$. (2) $B=0$. To see this, it is sufficient to consider the example of $S^1\times RP^3$ once again, where we recall that $\mathrm{Ker}(i_{\#})=0$ because $\mathrm{Sq}^2\circ\pi_2=0$. Let f, g be the functions defined above. Then $f^*(W_1^3)\cup g^*(W_1)=|y_1|_2\otimes W_1^3$ which is the generator of $H^4(S^1\times RP^3, Z_2)$, whilst $f^*(W_1)\cup g^*(W_1^3)$ and $f^*(W_1^2)\cup g^*(W_1^2)$ are trivial. Therefore, since we established above that $[S^1\times RP^3, RP^3]$ is Abelian, $\gamma_{\#}(|(f, g)|)=i_{\#}(B\cdot(f^*(W_1)^3\cup g^*(W_1)))=0$ implies that $B=0$. (3) $A=0$. To demonstrate this, we consider the example of $RP^2\times RP^2$, first noting that $[RP^2\times RP^2, RP^3]$ is an Abelian group. The latter follows from the fact that $\mathrm{Sq}^2\circ\pi_2=0$ which, together with $H^3(RP^2\times RP^2, Z)=0$, implies that $i_{\#}$ is an isomorphism of $H^4(RP^2\times RP^2, Z_2)$ onto $[RP^2\times RP^2, S^3]$ and therefore $[RP^2\times RP^2, RP^3]$ is a group extension of Z_2 by $Z_2\oplus Z_2$. But then the extension must be Abelian, for by Lagrange's theorem, it has order eight and must therefore be isomorphic to one of the five groups discussed in lemma 4. The only non-Abelian groups of order eight are D_4 and D_2^* which are extensions of Z_4 by Z_2. It therefore follows that $[RP^2\times RP^2, RP^3]$ is

Abelian. Consider the functions $f, g = k \circ p_1, k \circ p_2$ where p_1 and p_2 are the projection maps of $RP^2 \times RP^2$ and k is the inclusion $RP^2 \subset RP^3$. We have $k^*(W_1) = y_1$, the generator of $H^1(RP^2, Z_2)$ so that $f^*(W_1) = y_1 \otimes 1$ and $g^*(W_1) = 1 \otimes y_1$, and therefore $f^*(W_1^2) \cup g^*(W_1^2) = y_1^2 \otimes y_1^2$ which is the generator of $H^4(RP^2 \times RP^2, Z_2)$. Thus $(f, g)^*(|\theta|) = A \cdot (f^*(W_1)^2 \cup g^*(W_1)^2) = 0$ which is possible only if $A = 0$.

The combination of lemmas 1–7 establishes the main theorem.

Theorem 3. If X is a manifold of dimension $\leqslant 4$, $[X, RP^3]$ is an Abelian group.

5. Structure of $[X, S^3]$ for $\mathrm{Sq}^2 \circ \pi_2 = 0$

Recall that $[X, S^3]$ is a group extension of $H^4(X, Z_2) \cong Z_2$ by $H^3(X, Z)$. There is a decomposition of the latter group into its free and torsion subgroups:

$$H^3(X, Z) = Z^{b_3} \oplus \bigoplus_{k=1}^{n} Z_{m_k}$$

where b_3 is the third Betti number of X (non-zero if X is parallelisable) and the integers m_k satisfy the relation that m_k divides m_{k+1} for $1 \leqslant k \leqslant n-1$. The group of equivalence classes of Abelian extensions of Z_2 by $H^3(X, Z)$ may therefore be decomposed as

$$\mathrm{Ext}_Z(Z_2, H^3(X, Z)) \cong \mathrm{Ext}_Z(Z_2, Z)^{b_3} \oplus \left(\bigoplus_{k=1}^{n} \mathrm{Ext}_Z(Z_2, Z_{m_k}) \right).$$

The group $\mathrm{Ext}_Z(Z_2, Z)$ is trivial because any extension by a free Abelian group splits. $\mathrm{Ext}_Z(Z_2, Z_{m_k})$ is isomorphic to Z_2 with generator the class of the extension $Z_2 \rightarrowtail Z_{2m_k} \twoheadrightarrow Z_{m_k}$. Therefore Z^{b_3} will always split off any group $[X, S^3]$ although the torsion subgroup may involve non-trivial extensions. Thus $[X, S^3] \cong Z^{b_3} \oplus T'$ where T' may be a non-trivial extension of Z_2 by the torsion subgroup of $H^3(X, Z)$.

6. Structure of $[X, RP^3]$

By our above comments, $[X, S^3]$ will be of the form $Z^{b_3} \oplus T'$ where T' is either the torsion subgroup T of $H^3(X, Z)$ (if $\mathrm{Sq}^2 \circ \pi_2 \neq 0$), $Z_2 \oplus T$ or some non-trivial extension of Z_2 by T. $[X, RP^3]$ is a group extension of $[X, S^3]$ by $H^1(X, Z_2)$ which is isomorphic to Z_2^m for some $m \geqslant 0$, m being related to b_3. The group $[X, RP^3]$ is therefore an element of $\mathrm{Ext}_Z(Z^{b_3} \oplus T', Z_2^m)$ which may be decomposed as

$$\mathrm{Ext}_Z(Z, Z_2)^{mb_3} \oplus \mathrm{Ext}_Z(T', Z_2)^m.$$

Now $\mathrm{Ext}_Z(Z, Z_2)$ is isomorphic to Z_2, with generator the class of the extension $Z \rightarrowtail Z \twoheadrightarrow Z_2$, whilst $\mathrm{Ext}_Z(T', Z_2)$ is itself an extension of extension groups in general. It therefore follows that Z^{b_3} occurs as a direct summand in $[X, RP^3]$ in an extension of the type

$$Z^{b_3} \oplus Z_2^{m-k} \underset{I}{\rightarrowtail} Z^{b_3} \oplus Z_2^{m-k} \underset{\Phi}{\twoheadrightarrow} Z_2^k \oplus Z^{b_3-k} \oplus Z_2^{m-k}$$

where $I(g_i)$ is divisible by two in $[X, RP^3]$ for 'k' infinite cyclic generators g_i. Thus Z appears $m-k$ times as a direct summand and k times as $2 \cdot Z$ where the extension

$Z \rightarrowtail Z \twoheadrightarrow Z_2$ is regarded as $2 \cdot Z \subset Z \twoheadrightarrow Z_2$. To summarise, we have the following theorem.

Theorem 4. Let X be any four-manifold. Then the Abelian group $[X, S^3]$ is isomorphic to $Z^{b_3} \oplus T'$ where the torsion subgroup T' is isomorphic to: (i) The torsion subgroup of $H^3(X, Z)$ if X is non-compact or if X is compact with $\mathrm{Sq}^2 \circ \pi_2 \neq 0$, i.e. $[X, S^3] \cong H^3(X, Z)$; (ii) an extension of Z_2 by T if X is compact and $\mathrm{Sq}^2 \circ \pi_2 = 0$. Similarly, the Abelian group $[X, RP^3]$ is isomorphic to $Z^{b_3} \oplus Z_2^n \oplus T''$ where $m - b_3 \leqslant n \leqslant m$ for $m = \dim_{Z_2}(H^1(X, Z_2))$ and T'' is a group extension of T' by $H^1(X, Z_2)$.

7. Elementary examples

(1) $S^2 \times S^2$. The group $H^1(S^2 \times S^2, Z_2)$ is trivial, which implies that $[S^2 \times S^2, RP^3]$ is isomorphic to $[S^2 \times S^2, S^3]$. Moreover, because $H^3(S^2 \times S^2, Z)$ is also trivial, $[S^2 \times S^2, S^3]$ is determined by $\mathrm{Sq}^2 \circ \pi_2$ which is also trivial because it maps the generators $\langle S^2 \rangle \otimes 1$ and $1 \otimes \langle S^2 \rangle$ onto the classes $\langle S^2 \rangle^2 \otimes 1$ and $1 \otimes \langle S^2 \rangle^2$, which vanish because $\langle S^2 \rangle^2 = 0$. Hence $[S^2 \times S^2, RP^3] \cong [S^2 \times S^2, S^3] \cong H^4(S^2 \times S^2, Z_2) \cong Z_2$.

(2) CP^2. $H^1(CP^2, Z_2) = 0$ which implies that $[CP^2, RP^3] \cong [CP^2, S^3]$. Also, $H^3(CP^2, Z) = 0$ so that all depends upon the homomorphism $\mathrm{Sq}^2 \circ \pi_2$. $H^*(CP^2, Z)$ is the truncated polynomial algebra of height four generated by the class $C_1 \in H^2(CP^2, Z)$. Thus the top-dimensional cohomology group has generator C_1^2 and $|C_1|_2^2 = \mathrm{Sq}^2 \circ \pi_2(C_1)$ generates $H^4(CP^2, Z_2)$. It therefore follows that $i_{\#} = 0$, i.e. $[CP^2, S^3] \cong [CP^2, RP^3] = 0$.

(3) RP^4. $H^*(RP^4, Z_2)$ is the truncated polynomial algebra of height four on the generator $W_1 \in H^1(RP^4, Z_2)$. Also, $H^3(RP^4, Z) = 0$ which implies that $[RP^4, S^3]$ only depends on $\mathrm{Sq}^2 \circ \pi_2$. The group $H^2(RP^4, Z)$ is isomorphic to Z_2 with generator $x_2 = \beta_2(W_1)$ which satisfies $|x_2|_2 = \mathrm{Sq}^1(W_1) = W_1^2$. Thus $\mathrm{Sq}^2 \circ \pi_2(x_2) = W_1^4$ which is the generator of $H^4(RP^4, Z_2)$. This implies that $i_{\#} = 0$ and hence that $[RP^4, S^3] = 0$ and $[RP^4, RP^3] \cong H^1(RP^4, Z_2) \cong Z_2$.

(4) $RP^2 \times RP^2$. Recall from the proof of lemma 7 that $\mathrm{Sq}^2 \circ \pi_2 = 0$ so that $[RP^2 \times RP^2, S^3] \cong H^4(RP^2 \times RP^2, Z_2) \cong Z_2$. Also, $H^1(RP^2 \times RP^2, Z_2) \cong Z_2^2$ so that $[RP^2 \times RP^2, RP^3]$ is a group extension of Z_2 by Z_2^2, either Z_2^3 or $Z_4 \oplus Z_2$. Now $[RP^2, RP^3] \cong H^1(RP^2, Z_2) \cong Z_2$. Let $|k|$ be the generator. Then the map $z: H^1(RP^2 \times RP^2, Z_2) \rightarrow [RP^2 \times RP^2, RP^3]$ which sends the generators $y_1 \otimes 1$ and $1 \otimes y_1$ into $|k \circ p_1|$ and $|k \circ p_2|$ is a splitting homomorphism. z is a cross section because $W(|k \circ p_1|) = p_1^* \circ k^*(W_1) = p_1^*(y_1) = y_1 \otimes 1$ etc; it is a group homomorphism because $|k \circ p_1|^2 = |k^2 \circ p_1| = 0$ and similarly, $|k \circ p_2|^2 = 0$. Hence $[RP^2 \times RP^2, RP^3] \cong Z_2^3$.

(5) $S^1 \times S^3$. The group $H^2(S^1 \times S^3, Z)$ is trivial and therefore so is $\mathrm{Sq}^2 \circ \pi_2$. Also, $H^3(S^1 \times S^3, Z) \cong Z$ so that $[S^1 \times S^3, S^3]$ is the split extension $Z_2 \oplus Z$. $[S^1 \times S^3, RP^3]$ is therefore an extension of $Z_2 \oplus Z$ by $H^1(S^1 \times S^3, Z_2) \cong Z_2$. Let $|j|$ generate $\pi_1(RP^3) = Z_2$. Then $|j|^2 = 0$ and therefore the function $y_1 \otimes 1 \mapsto |j \circ p_1|$ is a splitting homomorphism and $[S^1 \times S^3, RP^3] \cong Z_2 \oplus Z \oplus Z_2$.

8. Spin properties of kink states

Shastri *et al* (1980) investigated the effect of 360° rotations about any spatial axis on kink states and obtained the following fermionic property for certain kink states.

Suppose that M is a compact, closed three-manifold of type 2, i.e. $[M, RP^3]$ is a split extension of $[M, S^3]$ by $H^1(M, Z_2)$ and that $|f| \in [M, S^3]$. Then if $|f|$ has odd kink number, $p_{4\#}(|f|) = f^*\langle S^3\rangle$ identified with deg(f) is odd, $|f|$ 'changes sign' under 360° rotations about any spatial axis. The effect of rotations on homotopy classes of maps is defined in terms of a map $\hat{\theta} : [M, S^3] \to [M, \Omega S^3] \cong [\Sigma M, S^3]$ and it is shown that if deg(f) is odd, $\hat{\theta}(|f|)$ is a non-trivial element of order two in $[\Sigma M, S^3]$. In the following sections, this construction will be generalised to arbitrary four-manifolds. Of course, the projection $p_{3\#}(|f|) \in H^3(X, Z)$ need no longer be labelled by an integer, but we still have a definition of even or odd parity via the coefficient homomorphism $\pi_{2*} : H^3(X, Z) \to H^3(X, Z_2)$, i.e. $f^*\langle S^3\rangle$ is even or odd according to $f^*\langle S^3\rangle_2$ zero or non-zero, and this definition reduces to the usual one for X homotopy equivalent to a three-manifold. Using this definition, we shall demonstrate that for any X with $\dim(X) \leq 4$ and $|f| \in [X, S^3]$ such that $f^*\langle S^3\rangle \neq 0 \bmod(2)$, $\hat{\theta}(|f|) \in [\Sigma X, S^3]$ is non-zero and of order two, a result which considerably generalises that of Shastri *et al.*

In order to carry out the calculations, we first have to obtain a level-five Postnikov resolution of S^3 which involves the calculation of the group $H^6(E_4, Z_2)$. However, in § 2 of this part of the paper is a definition of the map $\hat{\theta} : [X, S^3] \to [\Sigma X, S^3]$ which we factorise through $[X, RP^3 \wedge S^3]$ via a canonical map $\hat{\gamma} : RP^3 \wedge S^3 \to S^3$. After obtaining a level-five resolution of S^3 which involves knowledge of the groups $H^k(K(G, q), Z_2)$ for $G \cong Z$ or Z_2 and $0 \leq k \leq 7$, the map $\hat{\gamma}$ is analysed as a map $RP^3 \wedge S^3 \to E_5$ for E_5 five-equivalent to S^3. This enables us to calculate $\hat{\gamma}_\#$ and hence $\hat{\theta}$ and obtain the main theorem, theorem 5, on the effect of 360° rotations.

8.1. *Definition of* $\hat{\theta} : [X, S^3] \to [\Sigma X, S^3]$

The group SO(4) acts transitively on S^3 via the canonical action $\lambda : \mathrm{SO}(4) \times S^3 \to S^3$ where $\lambda : (\Lambda, x) \to \Lambda(x)$. The isotropy group of the group identity is the subgroup $\mathrm{SO}(3) \subset \mathrm{SO}(4)$ and, of course, SO(3) operates on S^3 via the map λ, the orbits of points x in S^3 with $x \neq \pm e$ being diffeomorphic to S^2. We therefore have a map $\lambda : \mathrm{SO}(3) \times S^3 \cong RP^3 \times S^3 \to S^3$ such that the related map γ defined by $\gamma(R, x) = x^{-1}\lambda(R, x) = x^{-1}R(x)$ is constant on the subspace $RP^3 \vee S^3$ and thus defines a map $\gamma : RP^3 \wedge S^3 \to S^3$ by $\hat{\gamma} \circ P = \gamma$ where P is the projection mapping $RP^3 \times S^3 \to RP^3 \wedge S^3 \equiv RP^3 \times S^3/(RP^3 \vee S^3)$. Suppose next that $R : S^1 \to RP^3$ is any parametrised rotation through 360° about a spatial axis. Then there is an induced map $R \wedge f : S^1 \wedge X \to RP^3 \wedge S^3$ and we may define a mapping $\theta(f) : S^1 \wedge X \to S^3$ by $\theta(f) = \hat{\gamma} \circ (R \wedge f)$. Correspondingly, one may define

$$\hat{\theta} : [X, S^3] \to [\Sigma X, S^3],$$
$$\hat{\theta} : |f| \mapsto |\theta(f)| = |\hat{\gamma} \circ (R \wedge f)| = \hat{\gamma}_\#(|(R \wedge f)|).$$

We therefore have a pairing, $\pi_1(RP^3) \times [X, S^3] \to [\Sigma X, S^3]$ which details the action of rotation loops on kink states and factors through the maps $|f| \mapsto |(R \wedge f)| \in [\Sigma X, RP^3 \wedge S^3]$ and a map $\hat{\gamma}_\# : [\Sigma X, RP^3 \wedge S^3] \to [\Sigma X, S^3]$. We will analyse $\hat{\gamma}_\#$ through a level-five Postnikov resolution of S^3.

8.2. *Level-five Postnikov resolution of* S^3

In this section, we shall factorise the four-equivalence $q_4 : S^3 \to E_4$ which we used earlier through a space E_5 which is five-equivalent to S^3:

$$\begin{tikzcd} F \arrow[r] \arrow[d, "v"', "q_4"] & S^3 \arrow[d, "q_5"] \arrow[dd, bend left, "q_4"] \\ K(Z_2, 5) \arrow[r, "j"'] & E_5 \arrow[d, "p_5"] \\ & E_4 \arrow[r, "\theta_4"'] & K(Z_2, 6). \end{tikzcd}$$

As usual, E_5 is the total space of the pull-back to E_4 of the path fibration over $K(Z_2, 6)$ (where we note that $\pi_5(S^3) \cong Z_2$) along a cohomology class $\theta_4 \in H^6(E_4, Z_2)$. Now given any class θ_4, q_4 lifts to $E_5(\theta_4)$ because $H^6(S^3, Z_2) = 0$. It also follows that the map q_5 associated with any non-trivial class θ_4 is a five-equivalence because the transgression $H^5(F_{q_4}, Z_2) \rightarrow H^6(E_4, Z_2)$ is an isomorphism. Thus knowledge of $H^6(E_4, Z_2)$ will enable us to construct a level-five Postnikov resolution. We can compute the latter group by examining the Serre exact sequence extracted from the spectral sequence of the orientable fibration $E_4 \xrightarrow{p_4} K(Z, 3)$, which we recall is the pull-back to $K(Z, 3)$ of the path fibration over $K(Z_2, 5)$. There is the following cohomology ladder of Serre exact sequences:

$$\begin{tikzcd} H^5(Z_2, 4, Z_2) \arrow[r, "T_5"] \arrow[d, equal] & H^6(Z, 3, Z_2) \arrow[r, "p_4^*"] & H^6(E_4, Z_2) \arrow[r, "i^*"] & H^6(Z_2, 4, Z_2) \arrow[r, "T_6"] \arrow[d, equal] & H^7(Z, 3, Z_2) \\ H^5(Z_2, 4, Z_2) \arrow[r, "T'_5"'] & H^6(Z_2, 5, Z_2) \arrow[r] \arrow[u, "(\mathrm{Sq}^2 \circ \pi_2)^*"] & 0 \arrow[r] & H^6(Z_2, 4, Z_2) \arrow[r, "T'_6"'] & H^7(Z_2, 5, Z_2). \arrow[u, "(\mathrm{Sq}^2 \circ \pi_2)^*"'] \end{tikzcd}$$

The bottom line of the ladder is the Serre sequence of the path fibration over $K(Z_2, 5)$ and the trivial group represents $H^6(PK(Z_2, 5), Z_2)$. It therefore follows that the transgressions T' associated with the path fibration are isomorphisms and that the transgressions T are defined by $T_q = (\mathrm{Sq}^2 \circ \pi_2)^* \circ T'_q$. Thus transgression for the fibration $E_4 \rightarrow K(Z, 3)$ is essentially composition of a cohomology operation with $\mathrm{Sq}^2 \circ \pi_2$: $(\mathrm{Sq}^2 \circ \pi_2)^*: |g| \mapsto |g \circ \mathrm{Sq}^2 \circ \pi_2|$. A knowledge of the generators of $H^5(Z_2, 4, Z_2)$, $H^6(Z_2, 4, Z_2)$ and $H^6(Z, 3, Z_2)$ will enable us to define T_5 and hence express $H^6(E_4, Z_2)$ in terms of $H^6(Z_2, 4, Z_2)$. Using theorems of Serre (1972), we obtain the following generators:

(a) $H^5(Z_2, 4, Z_2) \cong Z_2$ with generator $\mathrm{Sq}^1: H^4(*, Z_2) \rightarrow H^5(*, Z_2)$,
(b) $H^6(Z_2, 4, Z_2) \cong Z_2$ with generator $\mathrm{Sq}^2: H^4(*, Z_2) \rightarrow H^6(*, Z_2)$,
(c) $H^6(Z, 3, Z_2) \cong Z_2$ with generator $\mathrm{Sq}^3 \circ \pi_2: H^3(*, Z) \rightarrow H^6(*, Z)$.

The transgression homomorphism T_5 is therefore defined by $\mathrm{Sq}^1 \mapsto \mathrm{Sq}^1 \circ \mathrm{Sq}^2 \circ \pi_2$. Now the Adem relations (Spanier 1966)

$$\mathrm{Sq}^i \circ \mathrm{Sq}^j = \sum_{k=0}^{[i/2]} \mathrm{Sq}^{i+j-k} \circ \mathrm{Sq}^k \left| \binom{j-k-1}{i-2k} \right| \mathrm{mod}(2)$$

(where $0 < i < 2j$ and $[i/2]$ is defined as the largest integer less than $i/2$) imply that $\mathrm{Sq}^1 \circ \mathrm{Sq}^2 = \mathrm{Sq}^3: H^3(*, Z_2) \rightarrow H^6(*, Z_2)$, i.e. $x_3 \mapsto |x_3|_2^2$. It therefore follows that T_5 is an isomorphism, and hence if we choose θ_4 as the inverse image of Sq^2 under i^*, we have a Postnikov resolution of S^3 to level five. (T_6 is trivial because $\mathrm{Sq}^4: H^3(*, Z_2) \rightarrow H^7(*, Z_2)$ is the trivial map.)

By combining the Puppe sequences for p_5 and p_4, we obtain the following pair of linked sequences which completely characterise homotopy classes of maps from five-manifolds into S^3. It is important to note that these are only exact sequences of sets and maps, although some objects do lie in the category of groups.

$$
\begin{tikzcd}
 & & & H^2(X,Z) \arrow[d, "\mathrm{Sq}^2\circ\pi_2"] & \\
 & & {[X,S^3]} \arrow[d, "q_{5\#}"] \arrow[dr, "q_{4\#}"] \arrow[ddr, "q_{3\#}"'] & H^4(X,Z_2) \arrow[d, "i_\#"] \arrow[dr, "\mathrm{Sq}^2"] & \\
{[X,\Omega E_4]} \arrow[r, "\Omega\theta_{4\#}"] & H^5(X,Z_2) \arrow[r, "i_\#"'] & {[X,E_5]} \arrow[r, "p_{5\#}"'] & {[X,E_4]} \arrow[r, "\theta_{4\#}"'] \arrow[d, "p_{4\#}"] & H^6(X,Z_2) \\
H^3(X,Z_2) \arrow[u, "\Omega i_\#"] \arrow[ur, "\mathrm{Sq}^2"'] & & & H^3(X,Z) \arrow[d, "\mathrm{Sq}^2\circ\pi_2"] & \\
 & & & H^5(X,Z_2) &
\end{tikzcd}
$$

We shall only be interested in the application of the above system in the case where X is the suspension of a four-manifold, that is, we replace X^5 by ΣX^4. In this case, the diagram simplifies a little in that $\mathrm{Sq}^2\circ\pi_2 : H^2(\Sigma X, Z)\to H^4(\Sigma X, Z_2)$ coincides with the operation $\mathrm{Sq}^2\circ\pi_2 : H^1(X,Z)\to H^3(X,Z_2)$ under the suspension isomorphism because Steenrod squares commute with suspension. The latter cohomology operation is trivial. Similarly, the operations $\mathrm{Sq}^2\circ\pi_2$ and Sq^2 on $H^3(\Sigma X, Z)$ and $H^3(\Sigma X, Z_2)$ are the suspensions of the corresponding operations on $H^2(X,Z)$ and $H^2(X,Z_2)$. Finally, $H^6(Y,Z_2)$ is trivial for any five-manifold Y so that $p_{5\#}$ is a surjective map. We shall use the above resolution in the following sections in order to analyse the characteristic map $\hat{\gamma} : RP^3\wedge S^3\to S^3$.

8.3. *The characteristic map* $\hat{\gamma}$

We can now characterise the map $\hat{\gamma}$ up to level five by analysing the associated map $q_5\circ\hat{\gamma}$. In turn, $q_5\circ\hat{\gamma}$ is specified by $p_5\circ q_5\circ\hat{\gamma}=q_4\circ\hat{\gamma}$ from $RP^3\wedge S^3$ into E_4 which is specified by $p_4\circ q_4\circ\hat{\gamma}=q_3\circ\hat{\gamma}$, i.e. by the cohomology class $\hat{\gamma}^*\langle S^3\rangle\in H^3(RP^3\wedge S^3, Z)$. The latter group can be calculated by repeated application of the suspension isomorphism because $RP^3\wedge S^3$ is homeomorphic to $\Sigma^3 RP^3$, and hence

$$H^3(RP^3\wedge S^3, Z)\cong H^2(RP^3\wedge S^2, Z)\cong H^1(RP^3\wedge S^1, Z)\cong 0.$$

Therefore $p_{4\#}(|q_4\circ\hat{\gamma}|)=0$ which implies that $q_{4\#}(|\hat{\gamma}|)\in \mathrm{Im}(i_\#)$, that is, $q_{4\#}(|\gamma|)=i_\#(|\chi|)$ for $|\chi|\in H^4(RP^3\wedge S^3, Z_2)$. A string of suspension isomorphisms yields $H^4(RP^3\wedge S^3, Z_2)\cong H^1(RP^3, Z_2)\cong Z_2$. However, to obtain an explicit generator, we use the following lemma.

Lemma 8. $P^* : H^4(RP^3\wedge S^3, Z_2)\to H^4(RP^3\times S^3, Z_2)$ is an isomorphism.

Proof. This follows from the mapping sequence of the inclusion map $J : RP^3\vee S^3\subset RP^3\times S^3$ together with a homotopy equivalence of the mapping cone with the quotient space $RP^3\times S^3/(RP^3\vee S^3)\equiv RP^3\wedge S^3$ which yields the exact sequence

$$H^3(RP^3\times S^3)\xrightarrow[J_3^*]{} H^3(RP^3\vee S^3)\xrightarrow[\delta]{} H^4(RP^3\wedge S^3)\xrightarrow[P^*]{} H^4(RP^3\times S^3)\xrightarrow[J_4^*]{} H^4(RP^3\vee S^3).$$

Firstly, by the Mayer–Vietoris theorem, $H^4(RP^3 \vee S^3)=0$, so that P_4^* is an isomorphism. By exactness, $\mathrm{Ker}(P_4^*)=\mathrm{Im}(\delta)$. But δ is the trivial homomorphism (and hence P_4^* is an isomorphism) because J_3^* is an isomorphism. For, by the Kunneth theorem, $H^3(RP^3 \times S^3)$ has generators $W_1^3 \otimes 1$ and $1 \otimes \langle S^3 \rangle_2$ which are just $p_1^*(W_1^3)$ and $p_2^*(\langle S^3\rangle_2)$ where p_1 and p_2 are the projections of the product space. Therefore $J_3^* \circ p_1^*(W_1^3)=(p_1 \circ J)^*(W_1^3)=W_1^3$ and $J_3^* \circ p_2^*(\langle S^3\rangle_2)=\langle S^3\rangle_2$ because $p_i \circ J$ are identity maps. But W_1^3 and $\langle S^3\rangle_2$ are the generators of $H^3(RP^3 \vee S^3)$ by the Mayer–Vietoris theorem.

In the following, we shall identify the generator of $H^4(RP^3 \wedge S^3, Z_2)$ with its image $W_1 \otimes \langle S^3\rangle_2$ in $H^4(RP^3 \times S^3, Z_2)$. Recall that $q_{4\#}(|\hat{\gamma}|)=i_\#(|\chi|)$ for $|\chi| \in H^4(RP^3 \wedge S^3, Z_2)$. By the above lemma, we may write $|\chi|=C \cdot (W_1 \otimes \langle S^3\rangle_2)$ for some class C of Z_2, so that the mapping $\hat{\theta}:(X, S^3] \to [\Sigma X, S^3]$ is specified at level four by the class $q_{4\#} \circ \hat{\theta}(|f|)=q_{4\#}(|\hat{\gamma} \circ (R \wedge f)|)=|q_4 \circ \hat{\gamma} \circ (R \wedge f)|=i_\#((R \wedge f)^*(|\chi|))$. Thus by identifying $(R \wedge f)^*(|\chi|)$ with $(R \times f)^*(C \cdot (W_1 \otimes \langle S^3\rangle_2))=C \cdot (|y_1|_2 \otimes f^*(\langle S^3\rangle_2)$, where y_1 is the generator of $H^1(S^1, Z)$, we pinpoint the role of 'odd kink number'. This motivates the following definition.

Definition. A kink state $|f| \in [X, S^3]$ for $\dim(X) \leqslant 4$ has odd kink number if the associated class $\pi_{2*}(p_{4\#}(|f|))=f^*\langle S^3\rangle_2$ is non-zero.

Lemma 9. The universal obstruction C defined above is trivial.

Proof. Suppose that M is a compact three-manifold of type 2, (Shastri *et al* 1980) i.e. there exists a class $|f| \in [M, S^3]$ with degree $(f)=1$, implying that $f^*(\langle S^3\rangle_2)=\langle M\rangle_2$. Then according to Shastri *et al*, $q_{4\#}(\hat{\theta}(|f|))$ is non-trivial in $[\Sigma M, S^3]$. But then $C=1$, because $q_{4\#}(\hat{\theta}(|f|))=i_\#(C \cdot (W_1 \otimes \langle M\rangle_2))$ and $W_1 \otimes \langle M\rangle_2$ generates $H^4(\Sigma M, Z_2)$.

Corollary. If M is a three-manifold and $|f| \in [M, S^3]$ has odd kink number in the sense that $f^*\langle S^3\rangle_2=\langle M\rangle_2$, $\hat{\theta}(|f|)$ is non-zero and of order two.

Proof. Lemma 9 implies that $\hat{\theta}(|f|)=i_\#(|y_1|_2 \otimes \langle M\rangle_2) \neq 0$. We have already noted that $i_\#$ is a group homomorphism for four-manifolds. Thus $\hat{\theta}(|f|)$ has order two.

Corollary. If X is a four-manifold and $|f| \in [X, S^3]$ has odd kink number, $q_{5\#}(\hat{\theta}(|f|))$ is non-zero in $[\Sigma X, S^3]$.

Proof. $q_{5\#}(\hat{\theta}(|f|)) \neq 0$ because $q_{4\#}(\hat{\theta}(|f|))=p_{5\#} \circ q_{5\#}(\hat{\theta}(|f|)) \neq 0$.

Having established that $\hat{\theta}(|f|)$ is a non-trivial element of $[\Sigma X, S^3]$ for odd kink states, we now determine the order, again using the universal properties of $\hat{\gamma}$. Consider the class $\hat{\theta}(|f|)^2=|\theta(f)^2|=|\mathbf{2} \circ \hat{\gamma} \circ (R \wedge f)|$ where $\mathbf{2}: S^3 \to S^3$ is the squaring map. $\hat{\theta}(|f|)^2$ is specified by the map $\mathbf{2} \circ \hat{\gamma}: RP^3 \wedge S^3 \to S^3$ and it is clear that $q_{3\#}(|\mathbf{2} \circ \hat{\gamma}|)=2 \cdot \hat{\gamma}^*\langle S^3\rangle=0$. It therefore follows that $q_{4\#}(|\mathbf{2} \circ \hat{\gamma}|)=i_\#(|\hat{\chi}|)$ for some class $|\hat{\chi}| \in H^4(RP^3 \wedge S^3, Z_2)$. By our earlier remarks, there must exist a class D of Z_2 such that $|\hat{\chi}|=D \cdot (W_1 \otimes \langle S^3\rangle_2)$.

Lemma 10. The universal obstruction class D is trivial.

Proof. If $\dim(X) \leqslant 4$, $q_{4\#}(|\hat{\theta}(f)|)^2=i_\#(D \cdot (|y_1|_2 \otimes f^*\langle S^3\rangle_2))$. In particular, for $X=M$ a compact three-manifold of type 2, according to Shastri *et al* (1980), if $|f|$ has odd kink number, $q_{4\#}(|f|)$ is non-zero and of order two. This implies that $D = 0$.

By lemma 10, if $|f| \in [X, S^3]$, $\dim(X) \leq 4$ and $f^*\langle S^3\rangle_2 \neq 0$, $\hat{\theta}(|f|)$ is non-zero in $[\Sigma X, S^3]$ and $\hat{\theta}(|f|)^2$ is an element of $\mathrm{Ker}(p_{5\#})$. Therefore there must exist a class $\Phi \in H^5(\Sigma X, Z_2)$ such that $q_{5\#}(\hat{\theta}(|f|)^2) = j_{\#}(\Phi)$. This cohomology class is determined by $\hat{\gamma}$ since $q_{5\#}(|\mathbf{2} \circ \gamma|) \in \mathrm{Im}(j_{\#})$, that is, there exists a universal class $\hat{\Phi}$ such that $\Phi = (R \wedge f)^*(\Phi)$. Applying the usual string of suspension isomorphisms, we find that $H^5(RP^3 \wedge S^3, Z_2) \cong H^2(RP^3, Z_2)$ and the generator may be identified with the generator $W_1^2 \otimes \langle S^3\rangle_2$ of $H^5(RP^3 \times S^3, Z_2)$. This immediately implies the main theorem on spin.

Theorem 5. Let X be a manifold of dimension ≤ 4 with $H^3(X, Z_2) \neq 0$. Then if $|f| \in [X, S^3]$ has odd kink number, $f^*\langle S^3\rangle_2 \neq 0$, the class $\hat{\theta}(|f|) \in [\Sigma X, S^3]$ is non-zero and of order two.

Proof. By our earlier remarks, $f^*\langle S^3\rangle_2 \neq 0$ implies that $q_{5\#}(\hat{\theta}(|f|)) \neq 0$ and that $q_{5\#}(\hat{\theta}(|f|))^2 = j_{\#}((R \wedge f)^*(\hat{\Phi}))$. Now the universal class $\hat{\Phi}$ may be identified with $E \cdot (W_1^2 \otimes \langle S^3\rangle_2)$ for some class E of Z_2 and $(R \wedge f)^*(\hat{\Phi})$ may be identified with $E \cdot (|y_1|_2^2 \otimes f^*\langle S^3\rangle_2)$ which is trivial for *any* $|f|$ because $|y_1|_2^2 \in H^2(S^1, Z_2)$ is obviously trivial.

As a final remark, it is interesting to inquire about *even* kink states, i.e. $|f|$ with $f^*\langle S^3\rangle_2 = 0$. For, to obtain a boson-like interpretation, one should always have $\hat{\theta}(|f|) = 0$, expressing the invariance of even states under 360° rotations. This is certainly true for X homotopy equivalent to a three-manifold, because we have $q_{4\#}(\hat{\theta}(|f|)) = i_{\#}(|y_1|_2 \otimes f^*\langle S^3\rangle_2) = 0$, which implies that $\hat{\theta}(|f|) = 0$ since $q_{4\#}$ is a bijection for four-manifolds. However, if X is of dimension four one can only conclude that $q_{5\#}(\hat{\theta}(|f|)) \in j_{\#}(H^5(\Sigma X, Z_2))$ and there seems to be no elementary way of relating $|f|$ and the corresponding five-dimensional cohomology class. The result does hold for some four-manifolds, e.g. CP^2, but in the general case, the invariance remains to be demonstrated.

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